(*vi*) If $y(x) = e^x$, then find $J[e^x]$.

(*vii*) Define second variation of the functional $J[y]$.

(*viii*) Write the Fredholm integral equation of the second kind.

(*ix*) Write the volterra integral equation of the second kind.

(*x*) Find $L^{-1}\left\{\frac{1}{s(s^2+4)}\right\}$.

2. (*a*) Prove that

$$\int_0^\infty \frac{e^{-3t} - e^{-6t}}{t} dt = \ln 2.$$ 5

(*b*) Evaluate

$$L^{-1}\left\{\frac{s+1}{s^2+s+1}\right\}.$$ 5

Or

(*c*) Solve :

$$y'' + 2y' + 5y = e^{-t}\sin t\ ;\ y(0) = 0,\ y'(0) = 1.$$ 5

(*d*) Prove that Fourier transforms as a limit of Fourier series. 5

3. (*a*) Find the eigenvalues and eigenfunctions of the homogeneous integral equation :

$$g(s) = \lambda \int_1^2 \left[st + \left(\frac{1}{st} \right) \right] g(t)dt.$$ 5

(*b*) Solve the integral equation

$$g(s) = f(s) + \lambda \int_0^s e^{s-t} g(t)\, dt$$ and

evaluate the resolvent Kernel. 5

4. (*a*) Solve the Fredholm integral equation and evaluate the resolvent Kernel :

$$g(s) = 1 + \lambda \int_0^1 (1 - 3st) g(t) dt.$$ 5

(b) Solve the integral equation

$$g(s) = f(s) + \lambda \int_0^1 (s+t)g(t)\,dt$$

and find the eigenvalues. 5

5. (a) Find the distance between the parabola $y = x^2$ and the straight line $x - y = 5$. 5

(b) Find the variation of the functional

$$J[y] = \int_{-1}^{1} (y'e^y + xy^2)\,dx.$$ 5

6. Show that the functional

$$J[y(x)] = \int_0^1 [y(x) + 2y'(x)]\,dx$$

defined in the space $C_1[0, 1]$ in continuous on the function $y_0(x) = x$ in the sense of first-order proximity. 10

7. Find the fourier series of the periodic function

$$f(x) = x, \quad 0 < x < 2\pi.$$ 10

8. (*a*) Convert the initial value problem

$$y'' + \lambda s^2 y = 0;\ y'(0) = 0 = y(0)$$

into a Volterra integral equation. 5

(*b*) Solve the Abel integral equation :

$$f(s) = \int_0^s \left[g(t) | (s-t)^\alpha\right] dt,\ 0 < \alpha < 1.$$ 5
